Graphical Untyped Lambda Calculus Interactive Interpreter (GULCII)

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Outline

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Notation

Backslash is used for lambda:

- easier to type
- familiar from Haskell

Neighbouring lambdas can be combined with one backslash:

```
\a b s z . a s (b s z)
```

Corresponds to:

λa. λb. λs. λz. a s (b s z)

Church and Scott encodings

- encode data as lambda terms
- continuation passing style
- Church folds vs Scott case analysis

Continuation passing style

- the datum is a black box that knows itself
- the datum is passed functions that it calls with its deconstruction
- the datum has one argument per constructor
- each continuation has one argument per constructor argument

Simple types

Church and Scott encoding coincide for simple types.

Bool

Haskell:

```
data Bool
  = False
  | True
```

Church, Scott:

```
true  = \t f . t
false = \t f . f

and = \a b . a b a
or  = \a b . a a b
not = \a . a false true
```

The datum "true" takes two arguments, and returns the first, which (by convention) denotes the value True.

Pair

Haskell:

```
data Pair a b
  = Pair a b
```

Church, Scott:

```
pair = \a b p . p a b

fst  = \p . p (\a b . a)
snd  = \p . p (\a b . b)
```

The datum "pair x y" takes one argument, which is a function of two arguments, and passes it the stored values of "x" and "y".

Maybe

Haskell:

```
data Maybe a
  = Nothing
  | Just a
```

Church, Scott:

```
nothing = \n j . n
just    = \a n j . j a

maybe   = \n j m . m n j
```

Either

Haskell:

```
data Either a b
  = Left a
  | Right b
```

Church, Scott:

```
left   = \a l r . l a
right  = \b l r . r b

either = \l r e . e l r
```

Recursive types

- Church and Scott encoding differ for recursive types.
- Church encoding uses folds.
  The deconstruction continuation threads throughout the structure.
- Scott encoding is similar to case analysis.
  The deconstruction continuation unwraps one layer of constructors only.

Natural numbers

Haskell:

```
data Nat
  = Zero
  | Succ Nat
```

Church:

```
zero = \s z . z
succ = \n s z . s (n s z)
```

Scott:

```
zero = \s z . z
succ = \n s z . s n
```

Nat examples

Haskell:

```
Zero, Succ Zero, Succ(Succ Zero), Succ(Succ(Succ Zero))
```

Church, applying the same “s” “n” times at once:

```
\s z . z
\s z . s z
\s z . s (s z)
\s z . s (s (s z))
```

Scott, applying different “s” “n” times separately:

```
\s z . z
\s z . s (\s z . z)
\s z . s (\s z . s (\s z . z))
\s z . s (\s z . s (\s z . s (\s z . z)))
```

Church Nat succ

```
zero = \s z . z
succ = \n s z . s (n s z)

  succ zero
= {- definition of succ -}
  (\n s z . s (n s z)) zero
= {- beta -}
  \s z . s (zero s z)
= {- definition of zero -}
  \s z . s ((\s z . z) s z)
= {- beta -}
  \s z . s ((\z . z) z)
= {- beta -}
  \s z . s z
= {- definition of one -}
  one
```

Scott Nat succ

```
zero = \s z . z
succ = \n s z . s n

  succ zero
= {- definition of succ -}
  (\n s z . s n) zero
= {- beta -}
  \s z . s zero
= {- definition of zero -}
  \s z . s (\s z . z)
= {- definition of one -}
  one
```

Nat arithmetic

Church:

```
add = \m n . m succ n
mul = \m n . m (add n) zero
exp = \m n . n m
```

Scott, open terms with `letrec`:

```
add = \m n . m (\p . succ  (add p n)) n
mul = \m n . m (\p . add n (mul p n)) zero
exp = \m n . n (\p . mul m (exp m p)) one
```

Scott, closed terms with `fix`:

```
add = fix (\add . \m n . m (\p . succ  (add p n)) n)
mul = fix (\mul . \m n . m (\p . add n (mul p n)) zero)
exp = fix (\exp . \m n . n (\p . mul m (exp m p)) one)
```

Fixed point combinator

Semantics:

```
fix f = f (fix f)
```

Implementation:

```
\f . (\x . f (x x)) (\x . f (x x))
```

What it computes:

- The unique least fixed point under the definedness order.
- Allows recursive functions to be defined as closed terms.

Nat predecessor

Church (courtesy Wikipedia):

```
pred = \n f x . n (\g h . h (g f)) (\u . x) (\v . v)
```

Scott:

```
pred = \n . n (\p . p) zero
```

Nat conversion

Church arithmetic is more concise (and doesn't need `fix`).
Scott predecessor is comprehensible.
Mix and match?

```
churchToScott = \n . n scottSucc scottZero

scottToChurch = \n . n
    (\p . churchSucc (scottToChurch p))
    churchZero

scottToChurch = fix (\scottToChurch . \n . n
    (\p . churchSucc (scottToChurch p))
    churchZero)
```

Nat subtract and equality

Church:

```
sub = \m n . n pred m
```

Scott:

```
sub = \m n . m
  (\p . n (\q . sub p q) m)
  zero
equal = \m n . m
  (\p . n (\q . equal p q) false)
  (n (\q . false) true)
```

Unwrap a layer of constructor from each number and recurse.
There is a different `equal` for booleans:

```
equalBool = \a b . a b (not b)
```

List

Haskell:

```
data List a
  = Nil
  | Cons a (List a)
```

Church:

```
nil  = \c n . n
cons = \x xs c n . c x (xs c n)
```

Scott:

```
nil  = \c n . n
cons = \x xs c n . c x xs
```

List operations

Church, Scott:

```
isnil  = \l . l (\x xs . false) true
head   = \l . l (\x xs . x) error
```

Church (`tail` courtesy Wikipedia):

```
length = \l . l (\x xs . succ xs) zero
tail   = \l c n . l
    (\x xs g . g x (xs c)) (\xs . n) (\x xs . xs)
```

Scott:

```
length = \l . l (\x xs . succ (length xs)) zero
tail   = \l . l (\x xs . xs) nil
```

More Scott functions

```
compose = \f g x . f (g x)

fold = \f e l . l (\x xs . f x (fold f e xs)) e

sum  = fold add zero
ands = fold and true
ors  = fold or false

map = \f . fold (compose cons f) nil
all = \f . compose ands (map f)
any = \f . compose ors (map f)

take = \n l. n(\p. l (\x xs. cons x (take p xs))nil)nil
drop = \n l. n(\p. l (\x xs. drop p xs) nil) l

iterate = \f x . cons x (iterate f (f x))
```

Outline

How to perform lambda calculus

- single step graph reduction
- visualisation of current state
- sonification of changes in state
- open terms vs closed terms

Graph reduction

```
data Term
  = Free String
  | Reference Integer
  | Bound             -- de Bruijn index 0
  | Scope Term        -- see Lambdascope paper
  | Lambda Strategy Term
  | Apply Term Term

reduce
  :: Definitions    -- Map String  Term
  -> References     -- Map Integer Term
  -> Term
  -> Maybe (References, Term)
```

Three kinds of Lambda

- strict (syntax inspired by Haskell's -XBangPatterns):

  `(\v ! s) t`

  t is fully reduced before substitution into s.

- copy:

  `(\v ? s) t`

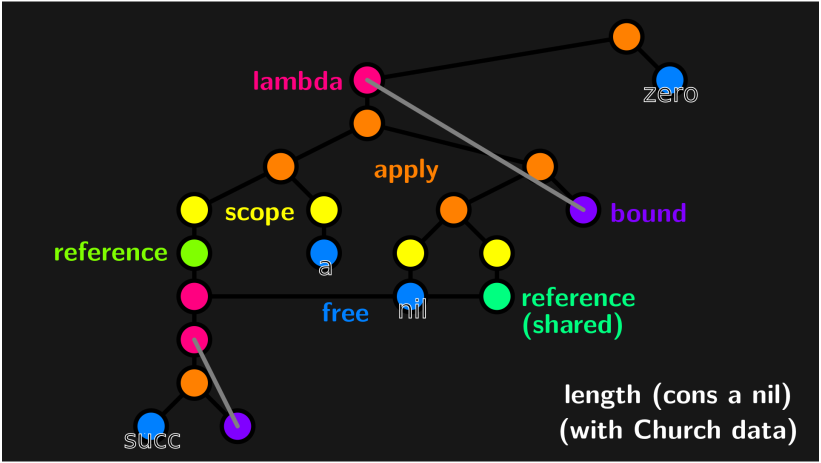
  t is substituted for each occurrence of v in s.

- lazy:

  `(\v . s) t`

  a new `Reference` is created for `t`, and substituted into s. `reduce` reduces *inside* the `References` until it is irreducible, at which point the `Reference` is replaced with the `Term` it refers to.

Visualisation



Sonification

- count number of nodes of each type
- statistics are forwarded to a Pure-data patch
- changes in each count control a harmonic (one for each type of node) in a simple phase modulation synth

Open terms

- free variables looked up on demand from environment
- allows definitions to be changed at runtime
- easier to write

Drawbacks of open terms

- no sharing
  subterms can be evaluated many times due to duplication
- exponential work (worst case)
- exponential space (worst case)

Fixed points

- closed terms with fixed point combinators
- allows evaluation to be shared
- sharing can be vital for efficiency

Outline

Better Evaluator

- current evaluator is still somewhat ad-hoc and doesn't preserve sharing
- previous evaluator even had correctness bugs
- switch to using Lambdascope (or similar) as a library?

Auto Fix

Automatically translating open terms to use fixed point combinators:

- recursive functions can use `fix`
- mutually recursive functions can use `fix` combined with tuples

```
many = some `orElse` none
some = one `andThen` many
```

becomes:

```
manysome = fix (\p -> pair
    (snd p `orElse` none)
    (one `andThen` fst p) )
many = fst manysome
some = snd manysome
```

Magic It

- refer to previously evaluated terms
- including the currently evaluating term
- without restarting evaluation

Haskell example (`ghci-8.0.1`):

```
> 3
3
> it + 5
8
> it * 2
16
```

Further Performances / Project Ideas

- "An infinite deal of nothing", a variety of non-terminating loops each with their own intrinsic computational rhythm.
- Implement in untyped lambda calculus an interpreter for a known Turing-complete tape mutation based language and run some simple programs in it.
  Illustrates Turing-completeness of untyped lambda calculus, albeit slowly.

Outline

Lambda calculus encodings

How to perform lambda calculus

Future work

EOF

EOF

Thanks!
Questions?

`https://mathr.co.uk`
`mailto:claude@mathr.co.uk`

`https://hackage.haskell.org/package/gulcii`
`https://code.mathr.co.uk/gulcii`